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Quantum Dynamics of Systems Connected by a Canonical Transformation

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Quantum Hamiltonian systems corresponding to classical systems related by a general canonical transformation are considered. The differential equation to find the unitary operator, which corresponds to the canonical transformation and connects quantum states of the original and transformed systems, is obtained. The propagator associated with their wave functions is found by the unitary operator. Quantum systems related by a linear canonical point transformation are analyzed. The results are tested by finding the wave functions of the under-, critical-, and over-damped harmonic oscillator from the wave functions of the harmonic oscillator, free-particle system, and negative harmonic potential system, using the unitary operator to connect them, respectively.

KEY WORDS: canonically connected; quantum systems; unitary operator; damped harmonic oscillator.

1. INTRODUCTION

The Hamiltonian formalism forms the basis for the structure of classical mechanics (Goldstein, 1988; Sudarshan and Mukunda, 1974) as well as providing a framework for theoretical extensions in many areas of physics. Especially, it gives much of the language from which quantum mechanics is constructed (Baym, 1969; Sakurai, 1994; Shankar, 1994). The canonical transformation that relates two canonical systems can provide a general procedure for readily solving classical systems (Goldstein, 1988; Sudarshan and Mukunda, 1974) and, in turn, understanding quantum systems. It forms a very large group that contains various types of subgroups determined by the description of the systems possessing

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geometrical and other kinds of symmetry (Goldstein, 1988; Sudarshan and Mukunda, 1974).

The classical system is determined by the classical equation of motion only, not the canonical momentum (Goldstein, 1988; Sudarshan and Mukunda, 1974). Thus, a single classical system has innumerable kinds of canonical variable sets, such that there are innumerable canonical transformations connecting any two classical systems. If one system is connected to another system by some canonical transformation, then we understand that it can be used to solve the other one easily, or it is a physically different system even though they are connected by a fixed mathematical relation. Some quantum systems have classical counterparts, and some do not, such as spin systems. When a physical system has no classical analogues, the structure of its Hamiltonian operator can be guessed, leading to results agreeing with empirical observation. When the physical system has classical analogues, we can recognize the structure of its Hamiltonian operator from the corresponding classical Hamiltonian. In this case, if we admit that the path integrals connect the quantum system and the classical system, then we can prove that the position and the momentum operators correspond to the canonical variables, and *vice versa* (Dittrich and Reuter, 1993; Feynman and Hibbs, 1965; Khandekar *et al.*, 1993; Schulman, 1981).

There are several questions regarding the connection between classical and quantum systems. The first is what is the connection between quantum counterparts of any two systems connected by the canonical transformation, if they indeed have quantum counterparts. The second is whether the quantum system corresponding to a classical system is unique. The third question is what is the quantum relation between two systems when they are physically different.

During the past few decades, there has been a surge of interest in the quantum mechanical solutions of a time-dependent oscillator system (Abdalla, 1987; Colegrave and Kheyrabady, 1986; Eckhardt, 1987; Gerry *et al.*, 1989; Hartley and Ray, 1982; Landovitz *et al.*, 1979; Lewis, 1967a,b; Lewis *et al.*, 1992; Lewis and Riesenfeld, 1969; Um *et al.*, 1987; Yeon *et al.*, 1987, 1993, 1997a,b, 1998, 2001). The Schrödinger equation of some systems cannot be solved easily due to mathematical difficulties. Thus, many authors have solved the time-dependent quantum system by their own specialized methods (Um *et al.*, 1987; Yeon *et al.*, 1987, 1993, 1997b, 2001). Lewis and Riesenfeld derived the dynamical invariant operator and solved the Schrödinger equation for time-dependent oscillators (Lewis, 1967a,b; Lewis *et al.*, 1992; Lewis and Riesenfeld, 1969). Some authors have chosen the operator methods to quantize the time-dependent oscillator system (Abdalla, 1987, Gerry *et al.*, 1989; Hartley and Ray, 1982). There are many other papers where time-dependent systems have been solved by different kinds of models (Colegrove and Kheyrabady, 1986; Eckhardt, 1987; Londovitz *et al.*, 1979). We also have published papers solying the time-dependent oscillator using path integrals and invariant operator methods (Um *et al.*, 1987; Yeon *et al.*, 1987, 1993, 1997b).

In a previous paper, we treated a general time-dependent quadratic Hamiltonian system. In that paper (Yeon *et al.*, 1997), we showed that the system can be quantized by invariant operator methods, and the quantum position and quantum momentum operators of that system correspond to classical canonical variables. We found the uncertainty relations for two system related by a linear gauge transformation. In another paper (Yeon *et al.*, 1998), we proved the quantum Hamiltonian is the same as the classical Hamiltonian whose canonical variables are replaced by the corresponding quantum operators, and we found the unitary operator corresponding to the linear canonical point transformation. The linear quantum gauge invariant and the scale transform were treated in that paper.

In this present paper, we find the unitary operator that is the quantum correspondence to the general classical canonical transformation, and we find the propagator related to the two wave functions of the canonically connected systems. We examine the physical meanings of two systems connected by a canonical transformation in the case when they are physically different and when they are treated as a single physical system.

In Sec. 2, we introduce a system and then find the Hamiltonian of the canonically connected system. We review the meaning of the canonical transformation. The Schrödinger equations of canonically connected systems can be obtained by using path integrals. We obtain the differential equation of the unitary operator connecting those Schrödinger solutions. We determine the relation of the quantum averages of a given operator in the system and the canonically transformed system. We find the propagator associated with the canonically connected systems. In Sec. 3, we introduce the general linear canonical point transformation and the systems connected by it. The quantum Hamiltonian of the canonical transformed system and the unitary operator are found, and we show that the relation of some variables between canonically connected systems is the same as the relation of the quantum averages of the operators corresponding to the variables. We obtain the uncertainty relations between the position and momentum operators in systems, that are canonically connected to each other.

In Sec. 4, we show some examples of the application of the quantum results obtained in Sec. 3. Section 4 helps to understand the results of the previous sections and presents the methods of quantization of the time-dependent system. We show that the harmonic oscillator, free-particle, and negative harmonic potential systems are classically connected with the under-, critical-, and over-damped harmonic oscillators by the linear canonical point transformation, respectively. The unitary operator, which is the quantum correspondence of the canonical transformation, can be obtained. The wave functions of the harmonic oscillator, free-particle, and negative harmonic potential systems are given first, and the exact wave functions of the under-, critical-, and over-damped harmonic oscillators are calculated by using the unitary operator, respectively. These results are the same as the previous results obtained by the operator method (Yeon *et al.*, 2001). Finally, in Sec. 5, we give the summary and conclusions.

2. QUANTUM CORRESPONDENCE OF THE GENERAL CANONICAL TRANSFORMATION AND THE CONNECTED CLASSICAL SYSTEMS

We first consider a system whose Hamiltonian is given by

$$H_1(q, p) = \frac{p^2}{2m} + V(q), \tag{1}$$

which gives the classical equation of motion. A general transformation of the variables (q, p) to other variables (Q, P) is taken as

$$\begin{cases} Q = Q(q, p, t) \\ P = P(q, p, t). \end{cases} \tag{2}$$

Let us assume that the inverse transformation of Eq. (2) exists and takes the form as

$$\begin{cases} q = q(Q, P, t) \\ p = p(Q, P, t). \end{cases} \tag{3}$$

If Eq. (2) and its inverse transformation, Eq. (3), are the canonical transformation, then a new Hamiltonian which gives (Q, P) can be obtained as

$$H_2(Q, P, t) = H_1(q, p, t) - p\frac{\partial q}{\partial t} - \frac{\partial F(Q, P, t)}{\partial t}, \tag{4}$$

where $F(Q, P, t)$ is a function which can be found by using Eq. (3) and the partial differential equations

$$P - p\frac{\partial q}{\partial Q} = \frac{\partial F(P, Q, t)}{\partial Q}, \tag{5}$$

$$-p\frac{\partial q}{\partial P} = \frac{\partial F(P, Q, t)}{\partial P}. \tag{6}$$

The new system (from now on, we will designate the new system as the Q-system and the old system as the q-system) represented by (Q, P) has two meanings. First, the Q-system is physically the same as the q-system. In this case, the Q-system is introduced only to solve the q-system, i.e., for mathematical convenience. Thus, the physical quantities represented in the Q-system are understood by their corresponding q-representation which is given by Eq. (2). Another case is that where Q-system is physically different from the q-system. In this case, the physical quantities represented in the Q-system are not related to the corresponding ones in the q-system, although they are mathematically connected to each other by Eq. (2).

From now on, in both cases, we treat the q- and Q-systems separately and consider the relations between the two systems quantum mechanically. Since the quantum mechanical wave function depends on the Hamiltonian as does the classical equation of motion, the mathematical forms of the wave functions of the

Q-system in both cases are the same, but the operators of the physical quantities are different. We assume that the quantum mechanical operators $(\hat{q}, \hat{p})$ and $(\hat{Q}, \hat{P})$ correspond to the classical canonical variables (q, p) and (Q, P) in the systems, respectively, that is,

$$[\hat{Q}, \hat{P}] = i\hbar, \tag{7}$$

$$[\hat{q}, \hat{p}] = i\hbar. \tag{8}$$

From the path integral methods, we obtain

$$i\hbar\frac{\partial\psi(q, t)}{\partial t} = \hat{H}_1\psi(q, t), \tag{9}$$

$$i\hbar\frac{\partial\Psi(Q, t)}{\partial t} = \hat{H}_2\Psi(Q, t). \tag{10}$$

When the highest degree of P in the Hamiltonian, Eq. (4), is 2, in Eqs. (9) and (10), the Hamiltonian operators each are of the same form as the classical Hamiltonians, Eqs. (1) and (4), whose canonical variables are replaced by the corresponding quantum operators.

We introduce the time-dependent unitary operator $U(\hat{q}, \hat{p}, t_2, t_1)$ connecting two wave functions which are solutions of Eqs. (9) and (10) as

$$\Psi(q, t_2) = U(\hat{q}, \hat{p}, t_2, t_1)\psi(q, t_1), \tag{11}$$

where this operator is defined as

$$U(\hat{p}, \hat{q}, t_2, t_1)\langle q| \equiv \langle q|\hat{U}(t_2, t_1). \tag{12}$$

The method to find the unitary operator is treated later in this section. From Eq. (11)

$$|\Psi(t_2)\rangle = \hat{U}(t_2, t_1)|\psi(t_1)\rangle. \tag{13}$$

Here, we know that $\hat{U}(t_2, t_1)$ has the property of unitarity. The relation of the quantum average of some observable operator $\hat{A}$ between the q- and Q-systems can be obtained as

$$\begin{aligned}\langle\hat{A}\rangle_\Psi &= \int \Psi^*(q)\hat{A}\psi(q)\,dq \\ &= \int \langle\psi|q\rangle U^\dagger(\hat{q}, \hat{p}, t_2, t_1)\hat{A}U(\hat{q}, \hat{p}, t_2, t_1)\langle q|\psi\rangle\,dq \\ &= \langle U^\dagger(\hat{q}, \hat{p}, t_2, t_1)\hat{A}U(\hat{q}, \hat{p}, t_2, t_1)\rangle_\psi,\end{aligned} \tag{14}$$

where $\langle\cdots\rangle_\Psi$ and $\langle\cdots\rangle_\psi$ are the quantum averages by the wave functions of the solutions of Eqs. (9) and (10), respectively. When both the q- and the Q-systems are physically the same and $\hat{A}$ is the quantum operator corresponding to the classical quantity A in the Ψ-space, $\hat{U}^\dagger(\hat{q}, \hat{p}, t_2, t_1)\hat{A}\hat{U}(\hat{q}, \hat{p}, t_2, t_1)$ is the quantum operator $\hat{A}$ in the ψ-space. If the q- and Q-system are considered physically different, $\hat{A}$ in the Ψ-space and $\hat{U}^{\dagger}(\hat{q}, \hat{p}, t_2, t_1)\hat{A}\hat{U}(\hat{q}, \hat{p}, t_2, t_1)$ in the ψ-space do not correspond to a classical quantity, although they have a close mathematical relation through Eq. (14).

Let us use $\hat{Q}$, $\hat{P}$ as operators in the Ψ-space and $\hat{q}$, $\hat{p}$ as operators in the ψ-space. That is, $\hat{Q}(\hat{Q}, \hat{P}, t)$ has to be averaged by $\Psi(Q, t)$ and $\hat{O}(\hat{q}, \hat{p}, t)$ has to be averaged by $\psi(q, t)$ only. Then, Eq. (14) can be expressed as

$$A(\hat{Q}, \hat{P}, t_2) = U^{\dagger}(\hat{q}, \hat{p}, t_2, t_1)\hat{A}(\hat{q}, \hat{p}, t_1)U(\hat{q}, \hat{p}, t_2, t_1), \tag{15}$$

and,

$$\hat{Q} = U^{\dagger}(\hat{q}, \hat{p}, t_2, t_1)\hat{q}U(\hat{q}, \hat{p}, t_2, t_1), \tag{16}$$

$$\hat{P} = U^{\dagger}(\hat{q}, \hat{p}, t_2, t_1)\hat{p}U(\hat{q}, \hat{p}, t_2, t_1). \tag{17}$$

From Eq. (13), we obtain

$$|\psi(t_1)\rangle = \hat{U}^{\dagger}(t_2, t_1)|\Psi(t_2)\rangle. \tag{18}$$

As we did above, we obtain

$$A(\hat{q}, \hat{p}, t_1) = U(\hat{Q}, \hat{P}, t_2, t_1)A(\hat{Q}, \hat{P}, t_2)\,U^{\dagger}(\hat{Q}, \hat{P}, t_2, t_1), \tag{19}$$

and

$$\hat{q} = U(\hat{Q}, \hat{P}, t_1, t_2)\,\hat{Q}U^{\dagger}(\hat{Q}, \hat{P}, t_1, t_2), \tag{20}$$

$$\hat{p} = U(\hat{Q}, \hat{P}, t_1, t_2)\,\hat{P}U^{\dagger}(\hat{Q}, \hat{P}, t_1, t_2). \tag{21}$$

Let us now find the unitary operator. The Schrödinger equations, Eqs. (9) and (10), can be written as

$$i\hbar\frac{\partial}{\partial t_1}|\psi(t_1)\rangle = \hat{H}_1|\psi(t_1)\rangle, \tag{22}$$

$$i\hbar\frac{\partial}{\partial t_2}|\Psi(t_1)\rangle = \hat{H}_2|\Psi(t_1)\rangle, \tag{23}$$

respectively. Substituting Eq. (13) into Eq. (23), we obtain

$$i\hbar\frac{\partial}{\partial t_2}\hat{U}(t_2, t_1) = \hat{H}_2\hat{U}(t_2, t_1), \tag{24}$$

and from Eqs. (18) and (22),

$$i\hbar\frac{\partial}{\partial t_1}\hat{U}^{\dagger}(t_2, t_1) = \hat{H}_1\hat{U}^{\dagger}(t_2, t_1). \tag{25}$$

If we set $t_1 = t_2$, Eqs. (22) and (23) give

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t, t) = \hat{H}_2(t)\hat{U}(t, t) - \hat{U}(t, t)\hat{H}_1(t). \tag{26}$$

From Eqs. (4), (20), and (21), the quantum Hamiltonian corresponding to $H_2(Q, P)$ becomes

$$\begin{aligned} H_2(\hat{Q}, \hat{P}, t) = {} & \hat{U}(t,t) H_1(\hat{Q}, \hat{P}, t) \hat{U}^\dagger(t,t) \\ & - \frac{1}{2} \hat{U}(t,t) \hat{P} \hat{U}^\dagger(t,t) \frac{\partial}{\partial t} (U(t,t) \hat{Q} U^\dagger(\hat{t}, t)) \\ & - \frac{1}{2} \frac{\partial}{\partial t} (U(t,t) \hat{Q} U^\dagger(\hat{t}, t)) \hat{U}(t,t) \hat{P} \hat{U}^\dagger(t,t) \\ & - \frac{\partial}{\partial t} F(\hat{Q}, \hat{P}, t). \end{aligned} \tag{27}$$

Substituting Eq. (27) into Eq. (26), we obtain the differential operator equation as

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \hat{U}(t,t) = - \Bigg(& \frac{1}{2} \hat{U}(t,t) \hat{P} \hat{U}^\dagger(t,t) \frac{\partial}{\partial t} (U(t,t) \hat{Q} \hat{U}(t,t)^\dagger(t,t)) \\ & + \frac{1}{2} \frac{\partial}{\partial t} U(t,t) \hat{Q} U^\dagger(t,t) \hat{U}(t,t) \hat{P} \hat{U}^\dagger(t,t) \\ & + \frac{\partial}{\partial t} F(\hat{Q}, \hat{P}, t) \Bigg) \hat{U}(t,t). \end{aligned} \tag{28}$$

Equations (13) and (18) give

$$\begin{aligned} \hat{U}(t_N, t_0) = {} & \hat{U}(t_N, t_{N-2}) \hat{U}^\dagger(t_{N-1}, t_{N-2}) \hat{U}(t_{N-1}, t_{N-4}) \\ & \times \hat{U}^\dagger(t_{N-3}, t_{N-4}) \cdots \hat{U}(t_{j+3}, t_j) \hat{U}^\dagger(t_{j+1}, t_j) \cdots \\ & \times \hat{U}^\dagger(t_4, t_3) \hat{U}(t_4, t_1) \hat{U}^\dagger(t_2, t_1) \hat{U}(t_2, t_0). \end{aligned} \tag{29}$$

For an infinitesimal time interval Δt, using Eqs. (24) and (25), we expand unitary operators $\hat{U}$ and $\hat{U}^\dagger$. If the terms of order $(\Delta t)^2$ or higher are ignored, Eq. (30) becomes

$$\begin{aligned} \hat{U}(t_N, t_0) & = \left(1 - \frac{i}{\hbar} \Delta t \hat{H}_2(t_{N-2})\right) \left(1 - \frac{i}{\hbar} 2\Delta t \hat{H}_2(t_{N-4})\right) \cdots \\ & \quad \times \left(1 - \frac{i}{\hbar} 2\Delta t \hat{H}_2(t_j)\right) \cdots \\ & \quad \times \left(1 - \frac{i}{\hbar} 2\Delta t \hat{H}_2(t_1)\right) \left(1 - \frac{i}{\hbar} 2\Delta t \hat{H}_2(t_0)\right) \hat{U}(t_0, t_0) \\ & = e^{-\frac{i}{\hbar} \int_{t_0}^{t_N} dt \hat{H}_2(t)} \hat{U}(t_0, t_0), \end{aligned} \tag{30}$$

where $\hat{U}(t_0, t_0)$ can be obtained from Eq. (28). Using the unit operator

$$\int dq|q\rangle\langle q| = 1 \tag{31}$$

and Eq. (13), we obtain

$$\begin{aligned}\Psi(Q, t) &= \langle Q|\Psi(t_2)\rangle \\ &= \int dq\langle Q|\hat{U}(t_2, t_1)|q\rangle\langle q|\psi(t_1)\rangle \\ &= \int dq K_{Q-q}(Q, t_2; q, t_1)\psi(q, t_1),\end{aligned} \tag{32}$$

where the propagator $K_{Q-p}(Q, t_2; q, t_1)$ yields the wave function in the Q-system from the wave function in the q-system. Using path integral methods, this is calculated as

$$\begin{aligned}K_{Q-q}(Q, t_2; q, t_1) &= \langle Q|\hat{U}(t_2, t_1)|q\rangle \\ &= K_Q(Q, t_2; q, t_1)U(\hat{q}, \hat{p}, t_1, t_1),\end{aligned} \tag{33}$$

where $K_Q(Q, t_2; q, t_1)$ is the propagator for the Q-system which is obtained by $H_2(\hat{Q}, \hat{P}, t)$.

Using the unit operator

$$\int dQ|Q\rangle\langle Q| = 1, \tag{34}$$

and Eq. (18), we obtain

$$\begin{aligned}\psi(q, t) &= \langle q|\psi(t_2)\rangle \\ &= \int dQ K_{q-Q}(Q, t_2; q, t_1)\Psi(Q, t_1),\end{aligned} \tag{35}$$

where $K_{q-Q}(Q, t_2; q, t_1)$ is the propagator which can yield the wave function in the q-system from the wave function in the Q-system. Using path integral methods, this is also calculated as

$$\begin{aligned}K_{q-Q}(Q, t_2; q, t_1) &= \langle q|\hat{U}(t_2, t_1)|Q\rangle \\ &= U^{\dagger}(\hat{Q}, \hat{P}, t_2, t_2)K_q(Q, t_2; q, t_1),\end{aligned} \tag{36}$$

where $K_q(Q, t_2; q, t_1)$ is the propagator for the q-system whose Hamiltonian is $H_1(\hat{q}, \hat{p}, t)$. So far in this section, we have discussed systematically the canonically connected quantum systems and their relations. If the relation between (q, p) and (Q, P), Eq. (2) or (3), is given exactly, we could be even more precise in our analysis above.

3. UNITARY TRANSFORMATION AND QUANTUM SYSTEMS CORRESPONDING TO CLASSICAL SYSTEMS CONNECTED BY A LINEAR CANONICAL POINT TRANSFORMATION

Using the results of Sec. 2, we now present quantum systems corresponding to classical systems connected by a linear canonical point transformation and the exact unitary operator between the quantum systems. In the q-system, the Hamiltonian is given by Eq. (1).

The general linear canonical point transformation between the canonical variables (q, p) and another set of canonical variables (Q, P), that is, the linear relation associated with Eq. (2), is

$$\begin{cases} Q = e^{\beta(t)}q \\ P = e^{-\beta(t)}p - \alpha(t)\, e^{\beta(t)}q, \end{cases} \tag{37}$$

and the inverse canonical transformation of Eq. (37), that is, the linear relation associated with Eq. (3), is

$$\begin{cases} q = e^{-\beta(t)}Q \\ p = e^{\beta(t)}(\alpha(t)Q + P), \end{cases} \tag{38}$$

where $\alpha(t)$ and $\beta(t)$ are real, arbitrary, and differentiable functions of t, respectively (from now on, for simplicity, we drop their time variable dependence except in special cases). Using Eqs. (4)–(6) with Eq. (38), the Hamiltonian which gives $Q(t)$ and $P(t)$ is written as

$$\begin{aligned} H_2(Q, P, t) = e^{2\beta}\frac{P^2}{2m} &+ \left(\dot{\beta} + e^{2\beta}\frac{\alpha}{m}\right) PQ \\ &+ \frac{1}{2}\left(e^{2\beta}\frac{\alpha^2}{m} + 2\alpha\dot{\beta} + \dot{\alpha}\right) Q^2 + V(Q), \end{aligned} \tag{39}$$

where $V(Q) = V(q)|_{q=e^{-\beta(t)}Q}$. The detailed form of Eq. (10) in the linear canonical point transformation is obtained as

$$\begin{aligned} i\hbar\frac{\partial\Psi(Q,t)}{\partial t} &= -e^{2\beta}\frac{\hbar^2}{2m}\frac{\partial^2\Psi(Q,t)}{\partial Q^2} \\ &\quad + \left(\dot{\beta} + e^{2\beta}\frac{\alpha}{m}\right)\left(\frac{1}{2}\Psi(Q,t) + Q\frac{\partial\Psi(Q,t)}{\partial Q}\right) \\ &\quad + \frac{1}{2}\left(e^{2\beta}\frac{\alpha^2}{m} + 2\alpha\dot{\beta} + \dot{\alpha}\right) Q^2\Psi(Q,t) + V(Q)\Psi(Q,t). \end{aligned} \tag{40}$$

The differential equation of the unitary operator, associated with Eq. (28), can be found as

$$i\hbar\frac{\partial}{\partial t}U = \dot{\beta}\left(Q\frac{\partial}{\partial Q} + \frac{Q}{2}\right)U + \left(\frac{\dot{\alpha}}{2} + \alpha\dot{\beta}\right)Q^2U, \tag{41}$$

where

$$U = U(\hat{Q}, \hat{P}, t, t) = \langle Q|\hat{U}(t, t)|Q\rangle. \tag{42}$$

It is easy to find the solution of Eq. (41) as

$$U(\hat{Q}, \hat{P}, t, t) = \exp\left[-\frac{i}{2\hbar}\alpha\hat{Q}^2\right]\exp\left[-\frac{i}{2\hbar}\beta(\hat{Q}\hat{P} + \hat{P}\hat{Q})\right]. \tag{43}$$

If $\psi(q, t)$ is a solution of the Schrödinger equation with the Hamiltonian of Eq. (1), we can prove that $U(\hat{q}, \hat{p}, t_2, t_1)\psi(q, t_1)$ is a solution of the Schrödinger equation, Eq. (40), by direct substitution. This is direct proof that the unitary operator satisfying Eq. (11) is Eq. (43). Using Eq. (43) and its complex conjugate, Eqs. (16), (17), (20), and (21) can be obtained as

$$\begin{cases} \hat{Q} = U^\dagger\hat{q}U = e^{\beta(t)}\hat{q} \\ \hat{P} = U^\dagger\hat{p}U = e^{-\beta(t)}\hat{p} - \alpha(t)\,e^{\beta(t)}\hat{q}, \end{cases} \tag{44}$$

and

$$\begin{cases} \hat{q} = U\hat{Q}U^\dagger = e^{-\beta(t)}\hat{Q} \\ \hat{p} = U\hat{P}U^\dagger = e^{\beta(t)}(\alpha(t)\hat{Q} + P). \end{cases} \tag{45}$$

Here, we know that Eq. (44) (or Eq. (45)) is of the same form as the canonical transformation, Eq. (37) (or Eq. (38)), whose canonical variables are replaced by the corresponding quantum operators. This means that the quantum mechanical transformation corresponding to the linear canonical point transformation, Eq. (37) (or Eq. (38)), is a unitary transformation by the unitary operator, Eq. (43). When the q-system is physically the same as the Q-system, and a given classical quantity A has the relation.

$$A(q, p, t) = B(Q, P, t), \tag{46}$$

where the quantum average of the operator $\hat{A}$ is

$$\langle A(\hat{q}, \hat{p}, t)\rangle_\psi = \left\langle B\left(e^{-\beta(t)}\hat{Q}, e^{-\beta(t)}[\alpha(t)\hat{Q} + \hat{P}], t\right)\right\rangle_\Psi. \tag{47}$$

However, $\langle\hat{B}(\hat{Q}, \hat{P}, t)\rangle_\Psi$ is not the quantum average of $\hat{A}$. In this case, the position and momentum operators are not $\hat{Q}$ and $\hat{P}$ but $\hat{q}$ and $\hat{p}$. Since the commutators $[\hat{q}, \hat{p}] = i\hbar$ and $[\hat{Q}, \hat{P}] = i\hbar$ hold, (q, p) and (Q, P) do not violate Heisenberg's uncertainty principle. Since $[q, Q] = 0$, the uncertainty between q and Q is 0. The operators $\hat{p}$ and $\hat{P} = e^{-\beta(t)}\hat{p} - \alpha(t)\,e^{\beta(t)}\hat{q}$ satisfy

$$[\hat{p}, \hat{P}] = i\hbar\,\alpha(t)\,e^{\beta(t)}, \tag{48}$$

and $\hat{q}$ and $\hat{P} = e^{-\beta(t)}\hat{p} - \alpha(t)\, e^{\beta(t)}\hat{q}$ satisfy

$$[\hat{q}, \hat{P}] = i\hbar\, e^{-\beta(t)}, \tag{49}$$

Thus, the uncertainty between $\hat{p}$ and $\hat{P}$ is

$$\sqrt{\langle(\Delta p)^2\rangle_\psi \langle(\Delta P)^2\rangle_\psi} \geq \frac{\hbar}{2}\alpha(t)\, e^{\beta(t)}, \tag{50}$$

and the uncertainty between q and P is

$$\sqrt{\langle(\Delta q)^2\rangle_\psi \langle(\Delta P)^2\rangle_\psi} \geq \frac{\hbar}{2}\, e^{-\beta(t)}, \tag{51}$$

where, for some operator,

$$\Delta x \equiv x - \langle x\rangle. \tag{52}$$

If the q-system is physically independent of the Q-system, and a given classical quantity A obeys the relation Eq. (46), the quantum average of $\hat{A}$ is given by Eq. (47) in the ψ-space becomes

$$\langle B(\hat{Q}, \hat{P}, t\rangle_\Psi = \left\langle A\left(e^{\beta(t)}\hat{q}, \left[e^{-\beta(t)}\hat{p} - \alpha(t)\, e^{\beta(t)}\,\hat{q}\right], t\right)\right\rangle_\psi, \tag{53}$$

in the Ψ-space. In this case, the uncertainty relation between $\hat{q}$ and $\hat{p}$ ($\hat{Q}$ and $\hat{P}$) satisfies the Heisenberg uncertainty principle in the ψ (Ψ)-space. Then, according to the above, $\hat{Q}$ and $\hat{P}$ in the ψ-space should be represented as $\hat{q}$ and $\hat{p}$ by Eq. (44), and $\hat{q}$ and $\hat{p}$ in the Ψ-space should be represented as $\hat{Q}$ and $\hat{P}$ by Eq. (45). With this consideration, the uncertainty relation between p and P in the ψ-space is Eq. (50), and that between q and P is Eq. (51). In the Ψ-space, the uncertainty relation between p and P can be calculated as

$$\sqrt{\langle(\Delta p)^2\rangle_\Psi \langle(\Delta P)^2\rangle_\Psi} \geq \frac{\hbar}{2}\alpha(t)\, e^{\beta(t)}, \tag{54}$$

and that between q and P can be calculated as

$$\sqrt{\langle(\Delta q)^2\rangle_\Psi \langle(\Delta P)^2\rangle_\Psi} \geq \frac{\hbar}{2}\, e^{-\beta(t)}. \tag{55}$$

4. APPLICATION OF PHYSICALLY SEPARATED QUANTUM SYSTEMS CONNECTED BY A LINEAR CANONICAL POINT TRANSFORMATION

If we know the wave function of the q-system, we can find the wave function of the Q-system. In this section we visualize the quantum results obtained in Sec. 3 with some examples, which show the methods of quantization of the time-dependent system. The harmonic oscillator, free-particle, and negative harmonic potential systems are connected to the under-damped harmonic oscillator, critically-damped harmonic oscillator, and over-damped harmonic oscillator, respectively. All of those are connected by a single linear canonical point transformation. The quantum systems connected by the linear canonical point transformation are physically different. Thus, for given wave functions of the harmonic oscillator, free-particle, and negative harmonic potential systems, wave functions of the under-damped, critically-damped, and over-damped harmonic oscillators are found by a unitary operator. Let us set the harmonic oscillator, free-particle, and negative harmonic potential systems as the q-system, and the under-damped, critically-damped, and over-damped harmonic oscillator systems as the Q-system. The Hamiltonian of the damped harmonic oscillator is

$$H_d = e^{-2\beta t}\frac{P^2}{2m} + \frac{m}{2}\omega_0^2\, e^{2\beta t} Q^2. \tag{56}$$

Comparing Eq. (56) with Eq. (39), for $H_2(Q, P, t)$, we see that

$$\alpha(t) = m\beta\, e^{2\beta t}, \tag{57}$$

$$\beta(t) = -\beta t, \tag{58}$$

$$V(Q) = \frac{m}{2}\omega^2\, e^{2\beta t} Q^2, \tag{59}$$

$$\omega^2 = \omega_0^2 - \beta^2. \tag{60}$$

Here, if $\omega^2 > 0$, $H_1(q, p, t)$ is the Hamiltonian of the harmonic oscillator potential; if $\omega^2 = 0$, it is the Hamiltonian of the free-particle; and if $\omega^2 < 0$, it is the Hamiltonian of the negative harmonic potential system. The canonical relation between the harmonic oscillator and the under-damped harmonic oscillator becomes

$$Q_{\mathrm{uh}} = e^{-\beta t}(A\,\sin\omega t + B\,\cos\omega t) = e^{-\beta t} q_{\mathrm{h}}, \tag{61}$$

where q_{h} is a general solution of the harmonic oscillator. The canonical relation between the free particle and the critically-damped harmonic oscillator becomes

$$Q_{\mathrm{ch}} = e^{-\beta t}(C_1 t + C_2) = e^{-\beta t} q_{\mathrm{f}}, \tag{62}$$

where q_{f} is a general solution of the free particle system. The canonical relation between the negative harmonic potential system and the over-damped harmonic oscillator becomes

$$Q_{\mathrm{oh}} = e^{-\beta t}(C_1 e^{\gamma t} + C_2 e^{-\gamma t}) = e^{-\beta t} q_{\mathrm{n}}, \tag{63}$$

where q_{n} is a general solution of the negative harmonic potential system and

$$\gamma = \sqrt{\beta^2 - \omega_0^2}. \tag{64}$$

The Schrödinger equation of the damped harmonic oscillator can be obtained as

$$i\hbar\frac{\partial\Psi(Q,t)}{\partial t} = -e^{-2\beta t}\frac{\hbar^2}{2m}\frac{\partial^2\Psi(Q,t)}{\partial Q^2} + \frac{m}{2}\omega_0^2 e^{2\beta t} Q^2\Psi(Q,t). \qquad (65)$$

This equation cannot be solved by the method of the separation of variables directly. We now find the solution of Eq. (65), which is given differently by the condition of ω_0 and β. Substituting Eqs. (57) and (58) into Eq. (43), the unitary operator can be found as

$$U(\hat{Q},\hat{P},t,t) = \exp\left[-i\frac{m\beta}{2\hbar}\, e^{2\beta t}\hat{Q}^2\right]\exp\left[i\frac{\beta}{2\hbar}t(\hat{Q}\hat{P}+\hat{P}\hat{Q})\right]. \qquad (66)$$

From Eq. (30), the position representation of the unitary operator between t_1 and t_2 becomes

$$U(\hat{q},\hat{p},t_1,t_2) = \langle q|\hat{U}(t_2,t_1)|q\rangle = e^{-\frac{i}{\hbar}\int_{t_1}^{t_2} dt H_2(\hat{q},\hat{p},t)}U(\hat{q},\hat{p},t_1,t_1). \qquad (67)$$

Substituting Eq. (67) into Eq. (11) and setting $t = t_1 = t_2$, from the wave function in ψ-space, the wave function in the Ψ-space, which is the solution of the Schrödinger equation in the Q-space, can be found as

$$\Psi(q,t) = U(\hat{q},\hat{p},t,t)\psi(q,t). \qquad (68)$$

The wave function of the harmonic oscillator is well known as

$$\psi_{\mathrm{h}}(q,t) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega(n+\frac{1}{2})t} H_n\left(\sqrt{\frac{m\omega}{\hbar}}q\right) e^{-m\omega q^2/2\hbar}. \qquad (69)$$

Substituting Eqs. (66) and (68), the wave function of the under-damped harmonic oscillator can be calculated as

$$\begin{aligned}\Psi_{\mathrm{uh}}(q,t) = &\left(\frac{\sqrt{m\omega/\pi\hbar}}{2^n n!}\right)^{1/2} e^{\frac{\beta}{2}t}\, e^{-i\omega(n+\frac{1}{2})t}\\ &\times H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, e^{\beta t}q\right) e^{-\frac{m\omega}{2\hbar}e^{2\beta t}q^2}\, e^{-\frac{i}{\hbar}\frac{m\beta}{2}e^{2\beta t}q^2},\end{aligned} \qquad (70)$$

where ω is given in Eq. (60). We can check that Eq. (70) is a solution of the Schrödinger equation of the damped harmonic oscillator, Eq. (65), in the case of $\omega_0^2 > \beta^2$ by direct substitution. The wave function of the free particle system is

$$\psi_{\mathrm{f}}(q,\lambda,t) = e^{-\frac{i}{\hbar}\lambda t}\left(C_1\, e^{\frac{i}{\hbar}\sqrt{2m\lambda}q} + C_2\, e^{-\frac{i}{\hbar}\sqrt{2m\lambda}\,q}\right), \qquad (71)$$

where C_1 and C_2 are arbitrary constants and λ is a continuous parameter having the dimension of energy. Substitution of Eqs. (66) and (71) into Eq. (68) gives the wave function of the critically-damped harmonic oscillator as

$$\Psi_{\rm ch}(q,\lambda,t) = e^{\frac{\beta t}{2} - i\frac{\lambda}{\hbar}t}\, e^{-ie^{2\beta t}\frac{m\beta}{2\hbar}q^2} \times \left(C_1\, e^{i\sqrt{\frac{2m\lambda}{\hbar^2}}e^{\beta t}q} + C_2\, e^{-i\sqrt{\frac{2m\lambda}{\hbar^2}}e^{\beta t}q} \right). \tag{72}$$

Here, we can check that Eq. (72) is a solution of the Schrödinger equation of the damped harmonic oscillator, Eq. (65), in the case of $\omega_0^2 > \beta^2$ by direct substitution. The wave function of the negative harmonic potential system is

$$\psi_{\rm nh}(q,\lambda,t) = e^{-\frac{i}{\hbar}\lambda t}\left[C_1 y_o\left(\sqrt{\frac{2m\gamma}{\hbar}}q,\, \frac{\lambda}{\gamma\hbar}\right),\, By_e\left(\sqrt{\frac{2m\gamma}{\hbar}}q,\, \frac{\lambda}{\gamma\hbar}\right)\right], \tag{73}$$

where $y_o(x,\eta)$ and $y_e(x,\eta)$ are

$$y_o(x,\eta) = x - \eta\frac{x^3}{3!} + \left(\eta^2 - \frac{3}{2}\right)\frac{x^5}{5!} + \left(-\eta^3 + \frac{13}{2}\eta\right)\frac{x^7}{7!} + \left(\eta^4 - 17\eta^2 + \frac{63}{4}\right)\frac{x^9}{9!} + \left(-\eta^5 + 35\eta^3\frac{531}{4}\eta\right)\frac{x^{11}}{11!} + \cdots, \tag{74}$$

$$y_e(x,\eta) = 1 - \eta\frac{x^2}{2!} + \left(\eta^2 - \frac{1}{2}\right)\frac{x^4}{4!} + \left(-\eta^3 + \frac{7}{2}\eta\right)\frac{x^6}{6!} + \left(\eta^4 - 11\eta^2 + \frac{15}{4}\right)\frac{x^8}{8!} + \left(-\eta^5 + 25\eta^3\frac{211}{4}\eta\right)\frac{x^{10}}{10!} + \cdots, \tag{75}$$

which are solutions of the differential equation (12),

$$\frac{d^2y}{dx^2} + \left(\eta + \frac{x^2}{4}\right)y = 0. \tag{76}$$

Substitution of Eqs. (66) and (73) into (68) gives the wave function of the over-damped harmonic oscillator as

$$\Psi_{\rm oh}(q,\lambda,t) = e^{-\frac{i}{\hbar}\lambda t}\, e^{\frac{\beta}{2}t}\, e^{-i\frac{m\alpha}{4\hbar}e^{2\beta t}q^2}\left\{C_1 y_o\left(\sqrt{\frac{2m\gamma}{\hbar}}\, e^{\beta t}q,\, \frac{\lambda}{\gamma\hbar}\right) + C_2 y_0\left(\sqrt{\frac{2m\gamma}{\hbar}}\, e^{\beta t}q,\, \frac{\lambda}{\gamma\hbar}\right)\right\}, \tag{77}$$

where γ is given in Eq. (64). Here, we can check that Eq. (77) is a solution of the Schrödinger equation of the damped harmonic oscillator, Eq. (65), in the case of $\omega_0^2 < \beta^2$ by direct substitution. The results of Eqs. (70), (72), and (77) have been obtained by the use of invariant methods (Yeon, *et al.*, 2001).

5. SUMMARY AND CONCLUSIONS

In this section, we summarize and discuss the results obtained in the previous sections. In Sec. 2, the general canonical transformation and the systems connected by it were introduced, and then the quantum correspondence was treated. We showed the Hamiltonian and general transformation of the coordinates and momenta between the old and new systems. We discussed the canonical transformation in terms of two physical cases. For one, the canonically transformed system is introduced as a mathematical convenience in order to solve the original system. The other is a canonically connected system physically different from the original system, although they are related by a mathematical relation.

The formal differential equation which determines the unitary operator connecting the quantum states of the original system and the canonical transformed system was derived. The explicit form of the differential equation of the operator depends on the relation of the variables between the canonically connected systems. In the case where the canonically transformed system is introduced for mathematical convenience, we explained that the quantum average of a given operator in the original quantum space and the quantum average of the similar transformation of that operator by the unitary operator in the unitary transformed quantum space are the same. In the case where the original and canonically transformed systems are physically different, we also explained that the same mathematical relation holds as above, but that is not the quantum average in unitary transformed systems. The propagator connecting the wave functions in the original and transformed spaces was evaluated. The propagator which enables one to obtain the wave function in the transformed space from that in the original space was found as the propagator of the transformed space multiplied by the unitary operator at constant time. The propagator which enables one to obtain that in the original space from that wave function in the transformed space was found as the Hermitian conjugate of the unitary operator at constant time multiplied by the propagator in the original space.

In Sec. 3, we presented quantum systems corresponding to the connected systems by the linear canonical point transformation in order to utilize the general treatment of Sec. 2. The general linear transformation whose linear coefficients are represented by two arbitrary functions was introduced, and the Schrödinger equation of the transformed system was determined by those coefficients. We found the unitary operator which connects two quantum states of the original and the transformed systems. We explained that in the case where the transformed system is introduced for solving the original system, the quantum averages of a physical quantity in both systems are related to each other, but in the case where the transformed system is separated from the original system, these are not related physically. We showed that the uncertainty between the position and the momentum operators in the original space and the transformed space satisfy Heisenberg's uncertainty principle, and the uncertainty between the position operators in both spaces is zero. The uncertainty between the position operators in the original space and the momentum operators in the transformed space depends on the coefficients of the linear canonical point transformation. The uncertainty between momentum operators in the original and the transformed spaces also depends on the coefficients of the linear canonical point transformation.

In Sec. 4, we treated some examples for the case where the original system and linear canonical point transformed system are physically separated. To compare the classical solutions, we showed that the harmonic oscillator and the under-damped harmonic oscillator, free-particle system and critical-damped harmonic oscillator, and the negative harmonic potential system and over-damped harmonic oscillator, are connected by the linear canonical point transformation, respectively. From the Hamiltonian of the transformed system, we found the coefficient functions of the linear canonical point transformation and the Schrödinger equation of the transformed system. The exact unitary operator connecting quantum states of both systems was obtained from the general form with those coefficients. From the wave functions of the harmonic oscillator, free-particle, and negative-harmonic potential system, we found the wave functions of the under-, critical-, and over-damped harmonic oscillators by that unitary operator.

In this paper, we presented the unitary transformation which corresponds to the classical general canonical transformation and the quantum systems that are connected by it. The systems connected by the transformation are physically viewed as either one system or different systems. The case where the transformed systems are physically different will be treated further in the future, because this can be used to solve difficult problems in quantum systems. In classical mechanics, we can solve easily an oscillating system using the Hamilton–Jacobi theory, which is a kind of canonical transformation (Goldstein, 1988; Sudarshan and Mukunda, 1974). There, the transformed Hamiltonian is constant, and its solution could be obtained by transforming to new canonical coordinates that are all cyclic. In the future, we plan to carry out a quantum treatment of the Hamilton–Jacobi theory by applying the method presented in this paper. There are several difficult problems, such as how to derive the Schrödinger equation of the transformed system and how to find the solution of the differential equation of the unitary operator, and so forth. We would also like to explore the solutions of a Morse oscillator, which of course is a considerably more complicated task than for the harmonic oscillator.

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